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A New Measure of Growth for Countable-Dimensional Algebras II

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A new dimension function on countable-dimensional algebras (over a field) was introduced in the first paper on this subject by means of certain infinite matrix representations. Here we show that its dimension values for finitely generated algebras exactly fill the unit interval [0, 1]. This suggests that the dimension is somewhat smoother than the GK-dimension. © 1995 Academic Press, Inc.

0. INTRODUCTION

A new concept of *bandwidth dimension* of countable-dimensional algebras (over a field) was introduced in [3] and [4] by means of certain infinite matrix representations. Bandwidth dimension provides a new measure of the growth of an algebra, which appears to be quite different from that given by the Gelfand–Kirillov dimension (GK-dimension). For instance, the free algebra $F\{x, y\}$ has the smallest possible bandwidth dimension (0), reflecting zero growth in terms of matrices, but the largest possible GK-dimension ($+\infty$) reflects exponential growth in terms of generators. The possible range of values for the GK-dimension of finitely generated algebras is known to be

$$0, 1, 2, \text{ any real } r > 2, \text{ and } +\infty.$$

(These were determined by Borho and Kraft in 1976 (for $r > 2$) and Bergman in 1978 (the initial values). See [2].) Our present paper is devoted to completely describing the possible range of values for the bandwidth dimension of finitely generated algebras. It culminates in the following theorem (announced in [3]), which suggests that the bandwidth dimension is somewhat "smoother" than the GK-dimension.

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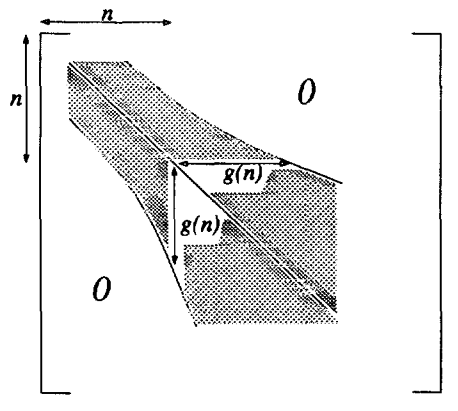
THEOREM 0.1 (The Principal Result). *For any field F, the bandwidth dimensions of finitely generated algebras over F exactly fill the unit interval* $[0, 1]$.

Let us now briefly recall the ideas and results in [4] which lead to the concept of bandwidth dimension. After that, in Section 1, we will outline our strategy for proving Theorem 0.1. The rest of the paper is then devoted almost entirely to the proof.

Firstly, by an *algebra* we shall always mean an associative algebra over a field, with an identity element. For any field F, the algebra of all $\omega \times \omega$ matrices over F which are simultaneously row-finite and column-finite is denoted by $B(F)$. For an element $x \in B(F)$ we say that a function g: $\mathbf{N} \to \mathbf{R}^+$ is a *growth curve* for x if for each $n \in \mathbf{N}$,

$$x(n,i) = 0 = x(i,n)$$

for all $i > n + g(n)$. In other words, $g(n)$ gives a bound on the "bandwidth" of x at the (n, n) position, if we interpret bandwidth as in the figure below.



We say that $x \in B(F)$ has *order at most $g(n)$ growth* (or that x has $O(g(n))$ growth) if there is a constant $c > 0$ such that the function $cg(n)$ is a growth curve for x. If A is a subalgebra of $B(F)$ and every $x \in A$ has $O(g(n))$ growth, then we say that the algebra A itself has $O(g(n))$ growth (but note that the constant c in $cg(n)$ will depend on the particular $x \in A$). To say A has *linear growth* means A has $O(n)$ growth.

The inspiration for looking at this type of matrix representation came from the result of Goodearl *et al.* [1, Proposition 2.1] that every countable-dimensional algebra A over a field F can be embedded in $B(F)$. In [4, Theorem 2.1] this was refined to say that A can be embedded in $B(F)$ as a subalgebra of linear growth. Thus A embeds in the algebra $G(1)$, where for a real number $r \in [0,1]$, $G(r)$ is the subalgebra of $B(F)$ given by

$$G(r) = \{x \in B(F) \quad x \text{ has } O(n^r) \text{ growth}\}.$$

It is then natural to look at the "least" r for which A embeds in $G(r)$, and call this the bandwidth dimension of A. Thus the *bandwidth dimension* of any countable-dimensional algebra A is defined to be

$$\inf\{r \in \mathbf{R},\ r \geq 0 \quad A \text{ embeds in } G(r)\}$$

or, equivalently,

$$\inf\{r \in \mathbf{R},\ r \geq 0 \quad A \text{ embeds in } B(F) \text{ with } O(n^r) \text{ growth}\}.$$

Note that by [4, Theorem 2.1] the bandwidth dimension of A is always in $[0,1]$. However, no examples were given in [4] where this dimension was other than 0 or 1. In our proof of Theorem 0.1 we will construct for each real number $r \in (0,1)$, a very explicit 8-generator subalgebra A of $G(r)$ of bandwidth dimension precisely r.

It is too early to say whether the bandwidth dimension of a general algebra A is easier or harder to compute than its GK-dimension. It seems likely that "most," if not all, algebras of finite GK-dimension will have zero bandwidth dimension. In that case, of course, bandwidth dimension may be the easier to compute when GK-dimension is finite, but the harder of the two when GK-dimension is infinite. We remark that the 8-generator algebra A above, of prescribed bandwidth dimension $r \in (0,1)$, does have infinite GK-dimension.

Finally, we give a word about our terminology. All rings and algebras are associative with an identity element, and all ring maps preserve the identity. The ground ring for our algebras is a field F. The ring of all $\aleph_0 \times \aleph_0$ column-finite matrices over F, with the rows and columns ordered in the standard way according to ω, is denoted by $M_\omega(F)$.

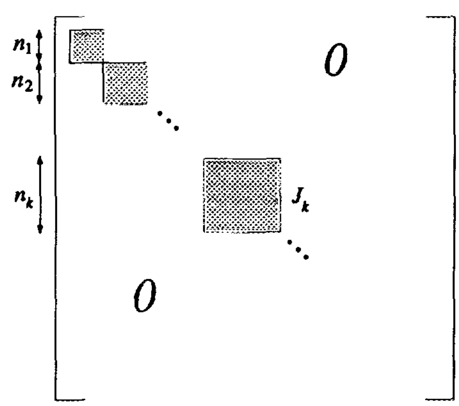
Throughout the paper, frequent use is made of the "big oh" notation: given two functions $f, g\colon \mathbf{N} \to \mathbf{R}^+$, to say $f(n)$ is $O(g(n))$ means that there is a positive constant c such that $f(n) \leq cg(n)$ for all $n \in \mathbf{N}$. In this situation we say "$f(n)$ has *order* at most $g(n)$."

1. The Strategy

Let r be a real number in $[0,1]$. We wish to construct a finitely generated algebra A of bandwidth dimension r. The case $r = 0$ is trivial.

By [4, Theorem 3.3], the case $r = 1$ is taken care of by any "purely infinite" A, that is, $A \cong A \oplus A$ as right A-modules. Therefore we can assume $0 < r < 1$.

The idea is to construct A as a "fat" subalgebra of $G(r)$. Being a subalgebra of $G(r)$ will certainly ensure that A has bandwidth dimension at most r. What we need to avoid is A being embeddable in $G(s)$ for some $s < r$. In particular, we cannot have $A \subseteq G(s)$ for any $s < r$. If we drop, for the moment, the requirement that A be finitely generated or even countable-dimensional, then we already have in [4] examples of algebras with this latter property, namely certain spines of $G(r)$. Recall from [4, Section 1] that the *spine* determined by an increasing sequence $n_1, n_2, \ldots, n_k, \ldots$ of positive integers is the natural copy S of $\prod_{k=1}^{\infty} M_{n_k}(F)$ inside $B(F)$, that is, S consists of all the block-diagonal matrices of the form



Let $t = r/(1 - r)$. By [4, Proposition 1.5], the spine S determined by the sequence $n_k = [k^t]$ fits inside $G(r)$ but not in any smaller $G(s)$. (Here [ ] denotes the integer part.) Even though S is not countable-dimensional, we can still talk of its bandwidth dimension (as an abstract algebra). Although at first sight this looks like it ought to be r, in fact the bandwidth dimension is 0 because S can be embedded in $G(s)$ for any $0 < s < r$, simply by "stretching" out the above representation and repeating blocks often enough (see [4, Corollary 6.5]). It can be shown that the algebra B generated by S and the standard one-dimensional infinite shifts $a, b \in B(F)$ is sufficiently "rigid" to have bandwidth dimension exactly r. The importance of the shifts is that they link successive blocks of the spine

S—primitive idempotents of one block become equivalent to those of the following block.

So how can we incorporate these features of B in a finitely generated algebra A (after all, B is uncountable-dimensional)? Although we can never have all of the spine S inside A, we can at least arrange for all its diagonal blocks J_k to be inside A. Then, as a measure of how "fat" A is, we can take the number of products of the generators of A that are needed to obtain each of the standard matrix units of J_k. It turns out that getting the matrix units in better than polynomial time (in k) is the key—our construction does it in $O((\log k)^2)$ products. (Throughout, logarithms are to base 2.) To keep A sufficiently rigid, we again need to ensure that A possesses appropriate links (cross-elements) from each block of the spine to the following block.

Our algebra A will be generated by eight elements of $G(r)$:

$$a, b, u, v, w, x, y, z.$$

The generators a and b are just the standard one-dimensional infinite shift matrices

$$a = \begin{bmatrix} 0 & & & & \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{bmatrix}.$$

Their principal role is to provide the links between successive blocks of the spine. The generators u, v, and w are designed to move one matrix unit of J_k to another very rapidly. The matrix u is a block diagonal matrix of strategically placed $2^m \times 2^m$ "binary search" matrices for various m, while v and w are certain "fast shift" matrices in $G(r)$. The generators x, y, and z are all in $G(0)$, and their role is to get *some* matrix unit of each J_k in sub-polynomial time.

Although we could work with the spine above, determined by $n_k = [k^t]$, it is more convenient to choose another spine of $G(r)$ whose block sizes are all powers of 2; it makes the bookkeeping easier when working with binary search matrices. The spine we use is described in the following proposition.

PROPOSITION 1.1 (Choice of the Spine S). *Let $0 < r < 1$ and let $t = r/(1-r)$. The sequence*

$$n_k = 2^{[t \log k]} \quad \textit{for } k = 1, 2, 3, \ldots$$

has the following properties:

(1) *The spine S of $B(F)$ determined by $n_1, n_2, \ldots$ is a spine for $G(r)$ but not for any smaller $G(s)$.*

(2) *Each $n_k = 2^{m_k}$ where $m_k = O(\log k)$.*

(3) *n_k has (at least) polynomial growth of k^t.*

(4) *For each $m \in \mathbf{N} \cup \{0\}$, let α_m be the number of blocks of size $2^m \times 2^m$ in the spine S. Then there is a constant $d \in \mathbf{N}$ such that $\alpha_m \leq 2^{dm}$ for all m. For all large m, we also have $m \leq \alpha_m$.*

Proof. Note that n_k is simply the largest integer power of 2 which is less than or equal to k^t. We know from [4, Proposition 1.5] that the spine determined by the sequence $\{[k^t]\}_{k=1}^{\infty}$ is a spine for $G(r)$ but not for $G(s)$ for any $s < r$. Since $n_k \leq [k^t] \leq 2n_k$, the same is true for the spine S in (1). It is clear that (2) and (3) hold.

Let $m \in \mathbf{N} \cup \{0\}$. If $n_k = 2^m$, then $k^t < 2^{m+1}$ and therefore $k < 2^{(m+1)/t}$. It is now obvious that there are at most $2^{(m+1)/t}$ positive integers k such that $n_k = 2^m$. Hence the existence of the constant d in (4) is clear.

Let $A(x) = 2^{x/t}$ and $B(x) = 2^{(x+1)/t}$ for all $x \in \mathbf{R}^+$. We have

$$\frac{B(x) - A(x)}{x} = \frac{(2^{1/t} - 1)2^{x/t}}{x} \to +\infty \qquad \text{as } x \to +\infty$$

because $t > 0$. Hence $B(x) - A(x) \geq 2x$ for all large x. In particular,

$$2^{(m+1)/t} - 2^{m/t} \geq 2m$$

for all large integers m. Therefore, for all large m, there are (by arguing very crudely!) at least m positive integers k satisfying

$$2^{m/t} \leq k < 2^{(m+1)/t},$$

that is, $n_k = 2^m$. Hence $m \leq \alpha_m$ for all large positive integers m. This establishes (4). ∎

2. Binary Search Matrices

The classical method of binary searching a sequential list of n items is based on "halving the search" at each step. It requires a search time of $O(\log n)$. The method suggests a matrix analogue. Let $n = 2^m$ for some $m \in \mathbf{N}$. In this section we define m "binary search" matrices in $M_n(F)$, which we show can be used to "find" any of the standard matrix units of $M_n(F)$ from any given one using $O(m) = O(\log n)$ multiplications. In the following section, one of the generators of our algebra A will be constructed as an infinite block diagonal matrix, where the blocks are strategically placed $2^m \times 2^m$ binary search matrices for various m.

DEFINITION 2.1. Let $m \in \mathbf{N}$. We define the $2^m \times 2^m$ *binary search* matrices $u_1, \ldots, u_m$ as follows:

$$u_1 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \qquad \text{in terms of } 2^{m-1} \times 2^{m-1} \text{ blocks}$$

$$u_2 = \begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix} \qquad \text{in terms of } 2^{m-2} \times 2^{m-2} \text{ blocks}$$

$$\vdots$$

$$u_i = \begin{bmatrix} 0 & I & & & & & \\ I & 0 & & & & & \\ & & 0 & I & & & \\ & & I & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & I \\ & & & & & I & 0 \end{bmatrix} \qquad \text{in terms of } 2^{m-i} \times 2^{m-i} \text{ identity blocks}$$

$$\vdots$$

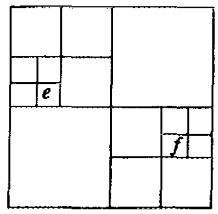
$$u_m = \begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix} \qquad \text{in terms of } 2^0 \times 2^0 \text{ blocks. } \blacksquare$$

PROPOSITION 2.2. *Let $n = 2^m$ for some $m \in \mathbf{N}$. Let $e, f \in M_n(F)$ be standard matrix units. Then f can be obtained from e by left and right multiplications by the $n \times n$ binary search matrices $u_1, \ldots, u_m$ and involving only $O(\log n)$ multiplications (in fact at most $2m$ multiplications).*

Proof. Observe that left (resp. right) multiplication of a matrix γ by u_i swaps blocks of 2^{m-i} rows (resp., columns) of γ in pairs. Hence starting with e, a left and/or right multiplication by u_1 produces a matrix unit whose nonzero entry 1 lies in the same $2^{m-1} \times 2^{m-1}$ block as that of f. (Of course, if e and f already have their 1's in the same $2^{m-1} \times 2^{m-1}$ block, then no multiplications by u_1 are required.) Now a left and/or right multiplication by u_2 produces a matrix unit with a 1 in the same $2^{m-2} \times 2^{m-2}$ block as that of f, and so on. For example, if $n = 2^3$, $e = e_{42}$, and

$f = e_{67}$, then

$$f = (u_2(u_1 e u_1) u_2) u_3.$$



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3. Scheme I Matrices and the Definitions of the Generators u, v, w

Recall from Section 1 that r is a fixed real number in $(0, 1)$, $t = r/(1-r)$, $n_k = 2^{[t \log k]}$ for each $k \in \mathbf{N}$, and S is the spine of $G(r)$ determined by $n_1, n_2, \ldots$. Also, for each $m \in \mathbf{N}$

$$\alpha_m = \text{number of blocks in } S \text{ of size } 2^m \times 2^m.$$

By a *scheme I matrix* we shall mean any block tridiagonal matrix in $B(F)$ determined by the spine S. Thus for such a matrix, its diagonal blocks appear in sequence as

$$\begin{array}{lcl} \alpha_0 & 1 \times 1 & \text{blocks} \\ \alpha_1 & 2 \times 2 & \text{blocks} \\ \vdots & & \\ \alpha_m & 2^m \times 2^m & \text{blocks} \\ \vdots & & \end{array}$$

Note that all scheme I matrices lie in $G(r)$ because $S \subseteq G(r)$ (Proposition 1.1).

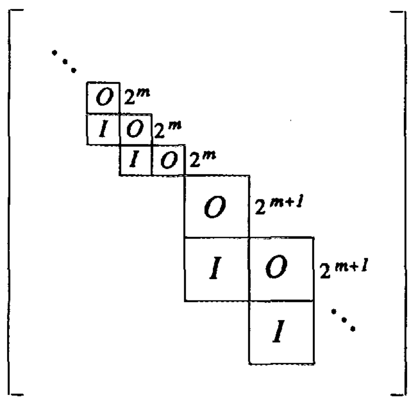
Definition 3.1. We define $u \in G(r)$ to be the element of the spine S whose 1×1 blocks are all 1, and whose $2^m \times 2^m$ blocks for each $m \in \mathbf{N}$ with $\alpha_m > 0$ consist of the $2^m \times 2^m$ binary search matrices $u_1, \ldots, u_m$ in groups of m, repeated as necessary to fill out all α_m blocks. The last group need not be a complete set.

The important property of u is that for any large m and any $2^m \times 2^m$ block position of S, there is some complete set of $2^m \times 2^m$ binary search matrices "not far away" on the diagonal of u. This is because $\alpha_m \geq m$ for large m (Proposition 1.1).

Next we define the generators v and w. They are the subdiagonal (respectively, superdiagonal) scheme I matrices displayed below. Essentially, v and w are the "fast shift" analogues of the one-dimensional shifts a and b, except that now v and w are not infinite shifts but rather "products of increasingly large finite shifts."

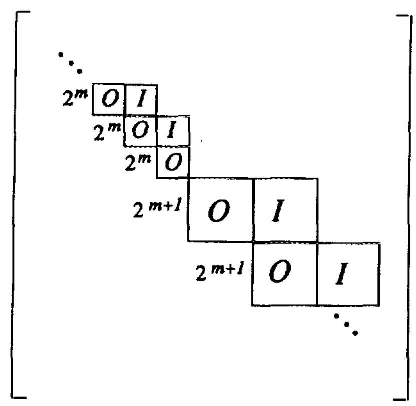
DEFINITION 3.2. Let v and w in $G(r)$ be the scheme I matrices

$v =$



and

$w =$



Of course, for some small m, it is possible that $\alpha_m = 0$, in which case we do not include any $2^m \times 2^m$ blocks for those m.

The generators v and w will enable us to move the $2^m \times 2^m$ binary search blocks of u rapidly back and forth along its diagonal. The shifts a and b, on the other hand, will be used more for "fine tuning."

4. Scheme II Matrices and the Definitions of the Generators x, y, z

Let $\beta_m = \alpha_m 2^m$ for all $m \in \mathbf{N} \cup \{0\}$. A *scheme II matrix* is any block diagonal matrix in $B(F)$ whose block sizes (in sequence) are

$$\beta_0, \beta_1, \beta_2, \ldots, \beta_m, \ldots .$$

The generators x, y, and z are certain scheme II matrices in $G(0)$—in fact, x and y have bandwidth 1 and z has bandwidth 0 (diagonal). (Actually u, v, and w are also scheme II matrices, but there is no advantage for the moment in viewing them as such.) Recall from Proposition 1.1 that there is a constant $d \in \mathbf{N}$ such that $\alpha_m \leq 2^{dm}$ for all m, while for all large m we also have $m \leq \alpha_m$. To cover this, and also a requirement later in the definition of z, we quantify "large" m.

Notation 4.1. Let m_0 be a fixed positive integer such that the following hold for all integers $m \geq m_0$:

(1) $m \leq \alpha_m \leq 2^{dm}$

(2) $dm + 1 + m \leq 2^m$.

Notation 4.2. For each integer $m \geq m_0$, we choose α_m distinct words

$$w_{m1}, w_{m2}, \ldots, w_{m\alpha_m}$$

of degree dm in two free variables. Note that this is possible because $\alpha_m \leq 2^{dm}$.

In order to define x and y, we require the following result which, in essence, is the construction used in [4] to get the free algebra $F\{x, y\}$ inside $G(0)$.

LEMMA 4.3. *Let $m \in \mathbf{N}$, $m \geq m_0$. For each $i = 1, 2, \ldots, \alpha_m$ there are matrices $x_{mi}, y_{mi} \in M_{dm+1}(F)$ of bandwidth* 1 *which satisfy*:

(1) $w_{mi}(x_{mi}, y_{mi}) = \begin{bmatrix} 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}$.

(2) $w_{mj}(x_{mi}, y_{mi}) = 0$ *for* $j \neq i$.

(3) $v(x_{mi}, y_{mi}) = 0$ *for any word v in two free variables and of degree greater than dm.*

Proof. This follows immediately from [4, Lemma 5.1] applied to the word w_{mi}. ∎

DEFINITION 4.4. Let x and y be the scheme II matrices

$$x = \begin{bmatrix} x_0 & & & & \\ & x_1 & & & \\ & & \ddots & & \\ & & & x_m & \\ & & & & \ddots \end{bmatrix}, \qquad y = \begin{bmatrix} y_0 & & & & \\ & y_1 & & & \\ & & \ddots & & \\ & & & y_m & \\ & & & & \ddots \end{bmatrix},$$

where for all $m < m_0$,

$$x_m = y_m = \text{the zero } \beta_m \times \beta_m \text{ matrix},$$

while for each $m \geq m_0$, x_m and y_m are the $\beta_m \times \beta_m$ matrices

$$x_m = \begin{bmatrix} \begin{bmatrix} x_{m1} & \\ & O \end{bmatrix} 2^m & & & \\ & \begin{bmatrix} x_{m2} & \\ & O \end{bmatrix} 2^m & & \\ & & \ddots & \\ & & & \begin{bmatrix} x_{m\alpha_m} & \\ & O \end{bmatrix} \end{bmatrix}, \qquad y_m = \begin{bmatrix} \begin{bmatrix} y_{m1} & \\ & O \end{bmatrix} 2^m & & & \\ & \begin{bmatrix} y_{m2} & \\ & O \end{bmatrix} 2^m & & \\ & & \ddots & \\ & & & \begin{bmatrix} y_{m\alpha_m} & \\ & O \end{bmatrix} \end{bmatrix}.$$

The definition of z is more straightforward. It is here that we use the condition (2) of Notation 4.1 in full.

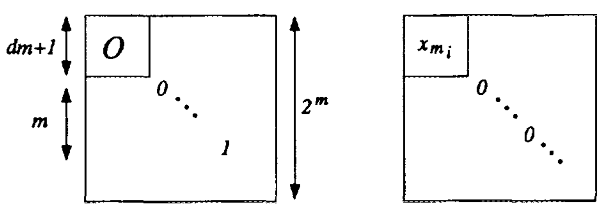
DEFINITION 4.5. Let z be the scheme II matrix

$$z = \begin{bmatrix} z_0 & & & & \\ & z_1 & & & \\ & & \ddots & & \\ & & & z_m & \\ & & & & \ddots \end{bmatrix}$$

where z_m is the zero $\beta_m \times \beta_m$ matrix for all $m < m_0$, while for each $m \geq m_0$ it is the $\beta_m \times \beta_m$ matrix

$$z_m = \begin{bmatrix} \boxed{1}\,2^m & & & \\ & \boxed{1}\,2^m & & \\ & & \ddots & \\ & & & \boxed{1} \end{bmatrix}$$

in which the single nonzero entry 1 in each $2^m \times 2^m$ diagonal block occurs at the $dm + 1 + m$ diagonal position within that block. The significance of this positioning is that it is exactly m positions down from the corresponding x_{mi} and y_{mi} blocks of x and y (see figure below).



5. OBTAINING THE MATRIX UNITS OF J_k IN LOGARITHMIC TIME

For each positive integer k, recall that J_k is the kth diagonal block of the spine S of $G(r)$ which we chose in Section 1. In particular, $J_k \cong M_{n_k}(F)$

for $n_k = 2^{[t \log k]}$. Our main objective in this section is to establish the following key property of our generators a, b, u, v, w, x, y, and z of A.

PROPOSITION 5.1 (The Key Property). *The algebra A contains each J_k, and the number of products of generators needed to obtain each of the standard matrix units in J_k grows essentially logarithmically in k—in fact, we can get by with $O((\log k)^2)$ products.*

As a first step towards the proof of 5.1, we have

LEMMA 5.2. *For each positive integer k, we can obtain some standard matrix unit of J_k in $O(\log k)$ products of the generators a, b, x, y, and z.*

Proof. Fix $k \in \mathbf{N}$. Then $n_k = 2^m$ for some $m \in \mathbf{N} \cup \{0\}$ with $m = O(\log k)$. There is no problem obtaining (albeit inefficiently) the general (i, j) matrix unit e_{ij} of $M_\omega(F)$ from just a and b, because $e_{ij} = a^{i-1}b^{j-1} - a^i b^j$. Therefore, to establish 5.2, we can ignore some small values of k and suppose $m \geq m_0$ (see 4.1). Suppose J_k occurs as the ith $2^m \times 2^m$ block of S where $1 \leq i \leq \alpha_m$, and let $w = w_{mi}$ be the corresponding word in two free variables of degree dm (see 4.2).

Claim. $w(x, y)b^m z a^m$ is the $(1, dm + 1)$ matrix unit of J_k.

Let $q = w(x, y)b^m z a^m$. To establish the claim, we shall view $w(x, y)$, $b^m z a^m$, and q as scheme II matrices, that is, block diagonal with block sizes $\beta_0, \beta_1, \ldots, \beta_n, \ldots$. Note that $b^m z a^m$ is just the diagonal matrix which results from z upon shifting its diagonal entries m places backwards.

For $n < m$, the $\beta_n \times \beta_n$ block of $w(x, y)$ is zero because $x_n = y_n = 0$ for $n < m_0$, and for $m_0 \leq n < m$

$$w(x_{nj}, y_{nj}) = 0$$

for $j = 1, \ldots, \alpha_n$. The latter is a consequence of Lemma 4.3(3) and the fact that $dm > dn$. Thus q has a zero $\beta_n \times \beta_n$ block for all $n < m$. This is also true for all $n > m$ because, on the typical jth $2^n \times 2^n$ diagonal sub-block, q is

$$\left[\begin{array}{cc} w(x_{nj}, y_{nj}) & \\ \hline & O \end{array}\right]\left[\begin{array}{c} O \\ \hline \begin{smallmatrix} 0 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} \end{array}\right] \begin{array}{l} \updownarrow\, dn+1 \\ \\ \updownarrow\, n-m \end{array}.$$

It remains to consider the $\beta_m \times \beta_m$ block of q. This comprises α_m diagonal $2^m \times 2^m$ blocks. On the typical jth such block for $j \neq i$, we see that $q = 0$ because

$$w(x_{mj}, y_{mj}) = 0$$

by Lemma 4.3(2). However, for $j = i$, the corresponding $2^m \times 2^m$ block is

$$\begin{array}{c} \\ dm+1 \updownarrow \end{array} \left[\begin{array}{ccc} 0 & \cdots & 1 \\ & \ddots & \\ & & 0 \\ \hline & & O \end{array}\right] \left[\begin{array}{ccc} 0 & & \\ & \ddots & \\ & & 1 \\ \hline & & O \end{array}\right] = \left[\begin{array}{ccc} 0 & \cdots & 1 \\ & \ddots & \\ & & 0 \\ \hline & & O \end{array}\right]$$

because of Lemma 4.3(1). Thus q is the $(1, dm + 1)$ matrix unit of J_k as claimed.

Finally, since w has degree dm, the product $w(x, y)b^m za^m$ involves

$$dm + m + 1 + m = (d + 2)m + 1 = O(m) = O(\log k)$$

occurrences of the generators a, b, x, y, and z. ∎

Proof of Proposition 5.1. Fix $k \in \mathbf{N}$. Again (as in Lemma 5.2) there is no loss of generality in assuming $n_k = 2^m$ where $m \geq m_0$. Our proof of Proposition 5.1 uses the generators u, v, and w for the first (and only) time. Their function is to position (efficiently) the $2^m \times 2^m$ binary search matrices into the kth diagonal position of the spine S.

Claim. Given any one of the $2^m \times 2^m$ binary search matrices $u_1, \dots, u_m$, there is an integer j with $1 \leq j \leq m - 1$ such that the matrix

$$v^j u w^j \qquad \text{or} \qquad w^j u v^j,$$

as a member of the spine S, has the given binary search matrix as its kth block.

To see this, first observe that because $m \geq m_0$, the generator u contains strings of *complete* sets of the $2^m \times 2^m$ binary search matrices $u_1, \dots, u_m$ along its diagonal (see Section 3). In fact, we can find a copy of each u_i at most $m - 1$ blocks away from J_k's position. Note that $v\,\delta w$ and $w\,\delta v$ are in S for all $\delta \in S$, so certainly $v^j u w^j$ and $w^j u v^j$ are in S for any $j \in \mathbf{N}$. Next observe what happens to a $2^m \times 2^m$ diagonal block γ of u when we form $v^j u w^j$. Provided γ is at least j positions from the end $2^m \times 2^m$ block of u, then γ gets shifted j blocks forward along the diagonal. A similar statement applies to $w^j u v^j$, where γ gets shifted j blocks backward. It is now evident that we can choose an integer j with the properties of the claim.

Using Lemma 5.2 we can obtain some standard matrix unit e of J_k in $O(\log k)$ products of generators of A. By Proposition 2.2 we know that, working inside J_k, we can obtain any other standard matrix unit f of J_k from e using $O(m)$ multiplications by the binary search matrices of J_k. But from the above claim, the effect of multiplying an element of J_k by a binary search matrix of J_k is the same as multiplying the element by a suitable $v^j uw^j$ or $w^j uv^j$ where $1 \leq j \leq m - 1$. (Just view the product inside the spine S.) So each of the $O(m)$ multiplications is replaced by another $O(m)$ multiplication by generators of A. Therefore we can reach f in $O(m^2) = O((\log k)^2)$ products of the generators a, b, u, v, w, x, y, and z. This completes the proof. ∎

Despite its importance, the property in Proposition 5.1 on its own is not enough to prevent A from being "stretched" under an embedding in $B(F)$. For although the generators a and b are not scheme II matrices, their role so far in Proposition 5.1 could equally well have been done by scheme II matrices (by dropping off the linking 1's between successive $\beta_n \times \beta_n$ blocks). Then all the generators of A would be scheme II, whence A would naturally sit inside $\Pi M_{\beta_n}(F)$ and so would have bandwidth dimension 0. To avoid any stretching of A, we need to ensure that there is an appropriate link between successive J_k's. The following concept turns out to be the link required.

Definition 5.3. For a general ring T and elements $\alpha, \beta \in T$, we coin the term *cross-element from βT to αT* to mean any $\gamma \in T$ such that $\gamma T = \alpha T$ and $T\gamma = T\beta$.

Note that if α and β are idempotents of T with $\beta T \cong \alpha T$, then any $\gamma \in \alpha T \beta$ which induces this isomorphism under left multiplication is a cross-element. In particular, the standard matrix unit e_{ij} of a ring $T = M_n(R)$ provides a cross-element from $e_{jj}T$ to $e_{ii}T$ (herein lies the importance of matrix units to us). Note too that a cross-element from βT to αT is automatically a cross-element from βU to αU for any overring U of T. In situations where the parent ring is not important, we sometimes speak of γ simply as a cross-element from β to α.

The next proposition formalizes the linking we require (later) between J_k and J_{k+1}.

Proposition 5.4. *Let $k \in \mathbf{N}$ and let f and g be any standard primitive idempotents of J_k and J_{k+1} respectively. Then A contains a cross-element from fA to gA which can be obtained in $O((\log k)^2)$ products of the generators of A.*

Proof. For each $i, j \in \mathbf{N}$ let e_{ij} be the standard matrix unit of $M_\omega(F)$ with a "1" in the (i, j) position. Then $f = e_{ii}$ and $g = e_{jj}$ for some positive integers i and j. Let n be the row index of the last row of J_k (that is, $n = n_1 + n_2 + \cdots + n_k$). By Proposition 5.1, we can obtain $e_{ni} \in J_k$ in $O((\log k)^2)$ products of generators of A, and we can get $e_{j,n+1} \in J_{k+1}$ also in $O((\log(k+1))^2) = O((\log k)^2)$ products. Since $ae_{ni} = e_{n+1,i}$ we have

$$e_{ji} = e_{j,n+1} a e_{ni} \in A.$$

By symmetry, $e_{ij} \in A$ also. Now $g = e_{ji}e_{ij}$, $e_{ji} = ge_{ji}$, $f = e_{ij}e_{ji}$, and $e_{ji} = e_{ji}f$ imply $gA = e_{ji}A$ and $Af = Ae_{ji}$. Thus e_{ji} is a cross-element from fA to gA which can be expressed in $O((\log k)^2)$ products of the generators of A. ∎

6. Growth Curves of Orthogonal Idempotents and Cross-Elements in $B(F)$

We leave our algebra A in this section to establish two general results in $B(F)$, which will play a crucial role in the proof of Theorem 0.1. They concern upper bounds, using growth curves, of two types. The first bound (Proposition 6.1) is on how far apart the first nonzero rows of two elements $\alpha, \beta \in B(F)$ can be, given that α, β, and some cross-element γ from β to α all have a common growth curve $f(n)$ which is increasing. The second bound (Proposition 6.3) is on the number of orthogonal idempotents of $B(F)$ which can have a common increasing growth curve $f(n)$ and for which each has some nonzero row within some specified range of rows.

Proposition 6.1. *Let $Q = M_\omega(F)$ and let $\alpha, \beta \in B(F)$ be nonzero. Suppose γ is a cross-element from βQ to αQ such that α, β, and γ all have a common growth curve $f(n)$ which is increasing. Let l (respectively m) be the row index of the first nonzero row of α (respectively, β). If $m \geq l$ then*

$$m - l \leq f(l) + f(l + [f(l)])$$

while if $m \leq l$ then

$$l - m \leq f(m) + f(m + [f(m)]).$$

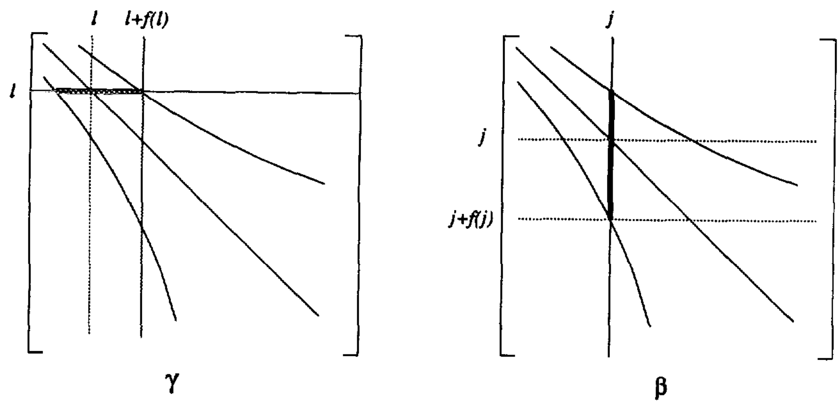
Proof. Suppose $m \geq l$. Since γ is a cross-element from βQ to αQ we have $\gamma Q = \alpha Q$, and so γ and α have nonzero rows in the same positions. Hence l is also the row index of the first nonzero row of γ. We now estimate how big m can be in terms of l.

Note first that we also have $Q\gamma = Q\beta$ and so γ and β have nonzero columns in the same positions. Since $f(n)$ is a growth curve for γ the nonzero entries in the lth row of γ must occur among the jth columns where $j \leq l + [f(l)]$. See the figure below. Hence β must have a nonzero jth column for some $j \leq l + [f(l)]$. But $f(n)$ is also a growth curve for β, and so for any j the nonzero entries in the jth column of β must occur in an ith row where $i \leq j + f(j)$. Since the mth row is the first nonzero row of β we thus have

$$\begin{aligned} m &\leq \max\{j + f(j) \colon j \leq l + f(l)\} \\ &\leq l + f(l) + f(l + [f(l)]) \end{aligned}$$

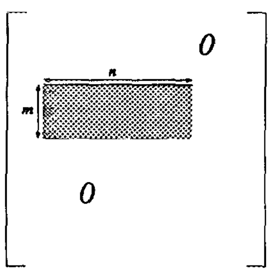
as required.

A similar proof works for the case $m \leq l$, where we would need the fact that $f(n)$ is a growth curve for α. ▮



Remark. Note that the proof of Proposition 6.1 uses both the upper and lower bounds provided by the growth curve (the upper curve is being used in the diagram for γ, while the lower curve is used in the diagram for β).

LEMMA 6.2. *Let m and n be positive integers and let Z be some fixed $m \times n$ rectangular block within $Q = M_\omega(F)$, that is, Z consists of all* *matrices whose nonzero entries are within the shaded area*:



(Note that we do not require any special placement of the rectangle.) Let $\{g_i\}_{i \in I}$ *be any set of orthogonal idempotents of* Q. *Then at most* n *of the* g_i *can have some nonzero row entirely within* Z.

Proof. Note that Z is a (unitary) left module over $B = M_m(F)$ of uniform (Goldie) dimension n. Suppose (after relabeling) that $g_1, g_2, \ldots, g_{n+1}$ each have a nonzero row entirely within Z, say

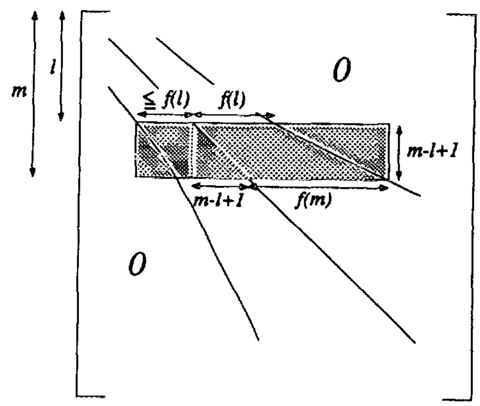
$$0 \neq h_i g_i \in Z$$

for a suitable (standard) primitive idempotent $h_i \in Q$. Then the $Bh_i g_i$ are nonzero independent left B-submodules of Z for $i = 1, \ldots, n+1$, which is impossible because ${}_BZ$ has uniform dimension n. ∎

PROPOSITION 6.3. *Let* $g_1, \ldots, g_k \in B(F)$ *be orthogonal idempotents and suppose* $f(n)$ *is an increasing growth curve for these idempotents. Suppose* l *and* m *are positive integers with* $l \leq m$ *and such that each* g_i *has some nonzero row between the* l*th and* m*th (inclusive). Then*

$$k \leq 2f(m) + m - l + 1.$$

Proof. By the very nature of a growth curve, all the g_i have some nonzero row entirely within the shaded area of the figure below.



Since $f(n)$ is increasing, this rectangular block has

$$\text{width} \leq f(l) + (m - l + 1) + f(m) \leq 2f(m) + m - l + 1.$$

Hence by Lemma 6.2, we have $k \leq 2f(m) + m - l + 1$. ∎

Remark. The above argument again utilizes the fact that $f(n)$ is both an upper and a lower growth curve.

7. Proof of Theorem 0.1

Let $0 < r < 1$, $t = r/(1 - r)$, and $n_k = 2^{[t \log k]}$, all as before. To complete the proof that our 8-generator algebra A has bandwidth dimension r, it suffices to show that if θ: $A \to G(s)$ is an algebra embedding for some $0 < s \leq r$, then $s = r$. Note that we have used the fact that $A \subseteq G(r)$ to infer that the bandwidth dimension of A is at most r. The only other properties of A we shall use for the remainder of the proof are the properties in Propositions 5.1 and 5.4. (Thus we no longer need to keep track of the exact nature of the generators of A.)

So let θ: $A \to G(s)$ be given. We shall show that $r \leq s$ in a series of steps. First we recall from [4] the definition of the subspaces $W_s(c)$ of $G(s)$. For each $c \geq 0$,

$$W_s(c) = \{x \in B(F) \ cn^s \text{ is a growth curve for } x\}.$$

Note that $G(s)$ is the union of these subspaces. It turns out that, because

$s < 1$, the powers of $W_s(c)$ grow "polynomially" (cf. the powers of $W_1(c)$, which grow exponentially). More precisely, by [4, Corollary 1.7] we have

Step 1. Let $c \in \mathbf{R}^+$. Then there is a positive constant d such that

$$(W_s(c))^m \subseteq W_s(dm^{1/1-s})$$

for all positive integers m.

Our second step uses the key property of A in Proposition 5.1, and its companion Proposition 5.4. For each $k \in \mathbf{N}$, let

$$F_k = \text{the set of standard primitive idempotents of } J_k.$$

Note that $|F_k| = n_k$ because $J_k \cong M_{n_k}(F)$.

Step 2. There exist increasing $c_k \in \mathbf{R}^+$ for $k = 1, 2, 3 \dots$ such that

(1) $c_k n^s$ is a growth curve for all elements in $\theta(F_k)$, for a cross-element between each pair, and also for a cross-element from any element of $\theta(F_k)$ to any one of $\theta(F_{k+1})$.

(2) $c_k = O((\log k)^{2/1-s})$.

Proof. Choose $c \in \mathbf{R}^+$ such that $W_s(c)$ contains the images (under θ) of the eight generators of A. This is possible because $\theta(A) \subseteq G(s) = \cup W_s(c)$ and the $W_s(c)$ form a chain. Let $k \in \mathbf{N}$. Let e, $f \in F_k$ and $g \in F_{k+1}$. By Propositions 5.1 and 5.4, we can obtain e, f, a cross-element from e to f, and a cross-element from f to g, all in $O((\log k)^2)$ products of generators of A. Their images under θ therefore all lie in $(W_s(c))^{d_k}$ where $d_k = O((\log k)^2)$. By Step 1,

$$(W_s(c))^{d_k} \subseteq W_s(c_k)$$

where $c_k = O(d_k^{1/1-s}) = O((\log k)^{2/1-s})$. This choice of c_k clearly satisfies (1) and (2). ▮

Our strategy now is to chase the first nonzero rows of the images of the idempotents in F_k, and obtain opposing constraints on their positions. On the one hand they have to be fairly close together to ensure that some cross-element from $\theta(F_k)$ to $\theta(F_{k+1})$ can lie in $W_s(c_k)$ (Step 3). On the other hand, having the n_k images of the equivalent orthogonal idempotents from F_k all inside $W_s(c_k)$ forces these nonzero rows to become increasingly scattered (Step 4).

For each $k \in \mathbf{N}$, we choose an idempotent $f_k \in F_k$ such that $\theta(f_k)$ has its first nonzero row at least as far down as for the images of any of the other $n_k - 1$ idempotents in F_k. We then set

$$p_k = \text{row index of the } \textit{first} \text{ nonzero row of } \theta(f_k).$$

Step 3. $p_k = O(k^{1/1-s} c_k^{1+s/1-s})$.

Proof. Let $m \in \mathbf{N}$. By Step 2 there is a cross-element γ from $\theta(f_m)$ to $\theta(f_{m+1})$ such that $\theta(f_m)$, $\theta(f_{m+1})$, and γ all have $f(n) = c_{m+1} n^s$ as a common growth curve. Hence, by Proposition 6.1,

$$\begin{aligned} p_{m+1} - p_m &\le f(p_m) + f(p_m + [f(p_m)]) \\ &\le 2f(p_m + [f(p_m)]) \\ &\le 2c_{m+1}(p_m + c_{m+1} p_m^s)^s \\ &\le 2c_{m+1}(1 + c_{m+1})^s p_m^s \qquad \text{since } s \le 1. \end{aligned}$$

Therefore, for all $m \in \mathbf{N}$ we have

$$p_{m+1} - p_m \le 2c_{m+1}(1 + c_{m+1})^s p_m^s. \tag{1}$$

Let $h\colon \mathbf{R}^+ \to \mathbf{R}^+$ be the function $h(x) = x^{-s}$ and note that h is a decreasing function since $s > 0$. By comparing the area under the graph of $h(x)$ with the area of the appropriate rectangle, we see from elementary calculus that for all $m \in \mathbf{N}$,

$$\frac{p_{m+1} - p_m}{p_m^s} \ge \int_{p_m}^{p_{m+1}} h(x)\, dx. \tag{2}$$

Hence for all integers $k \ge 2$ we have

$$\begin{aligned} \sum_{m=1}^{k-1} 2c_{m+1}(1 + c_{m+1})^s &\ge \sum_{m=1}^{k-1} \frac{p_{m+1} - p_m}{p_m^s} && \text{from (1)} \\ &\ge \sum_{m=1}^{k-1} \int_{p_m}^{p_{m+1}} h(x)\, dx && \text{from (2)} \\ &= \int_{p_1}^{p_k} h(x)\, dx = \frac{p_k^{1-s} - p_1^{1-s}}{1-s}. \end{aligned}$$

From this we get

$$\begin{aligned} p_k &\le \left(p_1^{1-s} + 2(1-s) \sum_{m=1}^{k-1} c_{m+1}(1 + c_{m+1})^s \right)^{1/1-s} \\ &\le \left(p_1^{1-s} + 2(1-s)(k-1) c_k (1 + c_k)^s \right)^{1/1-s} \end{aligned}$$

since the c_m are increasing. It is now clear that

$$p_k = O\left(k^{1/1-s} c_k^{1+s/1-s}\right). \quad \blacksquare$$

Step 4.

$$p_k \geq \frac{1}{(1+c_k)}\left(\frac{n_k}{5c_k}\right)^{1/s} \qquad \text{for all } k.$$

Proof. Fix a positive integer k. Let l be the smallest row index of any nonzero row of any of the images $\theta(g)$, as g ranges over the idempotents in F_k. Suppose this smallest index occurs for $\theta(g_k)$. Let $m = p_k$ ($\geq l$). By Step 2 there is a cross-element γ from $\theta(f_k)$ to $\theta(g_k)$ such that $\theta(f_k)$, $\theta(g_k)$, and γ all have $f(n) = c_k n^s$ as an increasing growth curve. Hence by Proposition 6.1,

$$\begin{aligned} m - l &\leq f(l) + f(l + [f(l)]) \\ &\leq f(m) + f(m + [f(m)]) \\ &\leq 2f(m + [f(m)]). \end{aligned}$$

By definitions of l and m, all the n_k images $\theta(g)$ of the idempotents $g \in F_k$ have a nonzero row between the lth and mth. Therefore by Proposition 6.3,

$$\begin{aligned} n_k &\leq 2f(m) + m - l + 1 \\ &\leq 2f(m) + 2f(m + [f(m)]) + 1 && \text{(from above)} \\ &\leq 5f(m + [f(m)]) \\ &\leq 5c_k(p_k + c_k p_k^s)^s \\ &\leq 5c_k(1 + c_k)^s p_k^s && \text{(since } s \leq 1) \end{aligned}$$

which implies (since $0 < s$) that

$$p_k \geq \frac{1}{1+c_k}\left(\frac{n_k}{5c_k}\right)^{1/s}.$$

∎

By comparing the contrasting estimates for the growth of the p_k which we have found in Steps 3 and 4, we obtain:

Step 5. $n_k = O(k^{s/1-s} c_k^{1+s/1-s})$.

Proof. From Step 3 there is a positive constant d such that

$$p_k \leq d k^{1/1-s} c_k^{1+s/1-s}$$

for all $k \in \mathbf{N}$. Comparing this with Step 4 we obtain

$$\frac{1}{(1+c_k)}\left(\frac{n_k}{5c_k}\right)^{1/s} \leq p_k \leq dk^{1/1-s}c_k^{1+s/1-s}$$

which implies

$$n_k \leq 5d^s k^{s/1-s}(1+c_k)^s c_k c_k^{(1+s/1-s)s}$$

and in turn that

$$n_k = O\left(k^{s/1-s}c_k^{1+s/1-s}\right). \quad \blacksquare$$

From Steps 2 and 5, we have

$$n_k = O\left(k^{s/1-s}(\log k)^q\right)$$

where $q = 2(1+s)/(1-s)^2$. But by Proposition 1.1, n_k has at least polynomial growth of k^t. Since polynomial growth always outstrips log growth, we must therefore have

$$t \leq s/(1-s).$$

Hence $r/(1-r) \leq s/(1-s)$ and so since $r, s \in (0,1)$ we must have $s \geq r$. This completes the proof of Theorem 0.1. $\blacksquare$

Remark. We can actually obtain a 2-generator algebra B of any prescribed bandwidth dimension $r \in [0,1]$ by taking $B = M_{10}(A)$, where A is our 8-generator algebra of bandwidth dimension r. This is because $M_{n+2}(C)$ is a 2-generator algebra for any n-generator algebra C (see [5, Lemma]), and forming a finite matrix algebra does not alter bandwidth dimension.

8. Bandwidth Dimension of Uncountable-Dimensional Algebras and Directly Finite Algebras

There are at least two approaches to the bandwidth dimension of uncountable-dimensional algebras A that one can adopt (see [4]). One way is simply to keep the definition

$$\inf\{r \in \mathbf{R},\ r \geq 0\ A \text{ can be embedded in } G(r)\}$$

but with the understanding that inf $\Phi = \infty$. An alternative approach is to mimic the GK-dimension and define the bandwidth dimension of A as the supremum of the bandwidth dimensions of its countable-dimensional subalgebras. In general, these two definitions are inequivalent, but by using either definition we have the following corollary to Theorem 0.1:

COROLLARY 8.1. *For each $r \in [0, 1]$ the algebra $G(r)$ has bandwidth dimension r.*

Proof. Certainly $G(r)$ has bandwidth dimension at most r. By the proof of Theorem 0.1, $G(r)$ contains a countable-dimensional subalgebra of bandwidth dimension r. Hence r is the supremum of the bandwidth dimensions of the countable-dimensional subalgebra of $G(r)$. On the other hand, if $G(r)$ itself were to embed in $G(s)$ for some $s < r$, this would imply that all the countable-dimensional subalgebras of $G(r)$ have bandwidth dimension less than r. Thus, with either definition, $G(r)$ has bandwidth dimension precisely r. ∎

It was claimed in Section 1 that if we take the spine S determined by the sequence $n_k = [k^t]$, where $t = r/(1 - r)$ and $0 < r < 1$, and let B be the subalgebra of $B(F)$ generated by S and the two infinite shifts a and b, then B has bandwidth dimension r. (In fact, we need only include the shift a.) Here we are using the first of the above definitions of bandwidth dimension. The proof of this claim is quite a bit simpler than the proof of Theorem 0.1, because it turns out that here we can choose all the c_k in Step 2 of Section 7 to be the same. The reason for this is that if R is any algebra isomorphic to $\prod_{i=1}^{\infty} M_{n_i}(F)$ for some unbounded increasing sequence $\{n_i\}$ of positive integers, and θ: $R \to G(s)$ is an algebra embedding, then the whole of the image $\theta(R)$ is inside some fixed growth curve $g(n) = cn^s$. In turn, the proof of this hinges on the self-injectivity of R and the fact that all the nonzero singular R-modules have uncountable dimension over F. (The author thanks K. R. Goodearl for supplying a proof of this last statement.)

In view of the fact that every countable-dimensional algebra has its bandwidth dimension in $[0, 1]$, the following is an immediate corollary of Theorem 0.1.

COROLLARY 8.2. *For any field F, the bandwidth dimensions of countable-dimensional algebras over F exactly fill the unit interval* $[0, 1]$.

Any countable-dimensional algebra A which is purely infinite, that is, $A_A \cong A \oplus A$, must have bandwidth dimension 1 by [4, Theorem 3.3]. Such algebras are, in particular, *directly infinite*—that is, they contain elements α, β with $\alpha\beta = 1$ but $\beta\alpha \neq 1$; equivalently, A_A is isomorphic to a proper direct summand of itself. For $r \in (0, 1)$, the countable-dimensional algebra A of bandwidth dimension r, which we constructed for the proof of Theorem 0.1, is also directly infinite because $ba = 1 \neq ab$. One might suspect, therefore, that having positive bandwidth dimension is somehow tied up with this type of skewness. However, our next result shows that this is not so.

PROPOSITION 8.3. *For any real number $r \in [0, 1]$ there is a countable-dimensional, directly finite algebra D of bandwidth dimension r. (By directly finite we mean all one-sided inverses are two-sided.)*

Proof. The case $r = 0$ is trivial. Suppose $0 < r < 1$. Let C be the subalgebra of $G(r)$ generated by

$$a, b_1, u, v, w, x, y, z$$

where a, u, v, w, x, y, z are the same generators as before but where b_1 is the scheme II matrix obtained from b by dropping off the linking 1's. That is, the $\beta_m \times \beta_m$ block of b_1 is

$$\begin{bmatrix} 0 & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ & & & & & 0 \end{bmatrix}.$$

Then, as remarked in Section 5, the proof of 5.1 still works for the algebra C. The difficulty occurs in 5.4. For although the same proof shows that, for $f = e_{ii} \in J_k$ and $g = e_{jj} \in J_{k+1}$, the element $e_{ji} \in C$, we cannot conclude that e_{ji} is a cross-element from fC to gC. It is true that e_{ji} is a cross-element from f to g in the larger ring $B(F)$, but this might not be so for their images under an embedding θ: $C \to G(s)$ (which is what is required in Step 2 in Section 7).

To get around this difficulty, let D be the subalgebra of $G(r)$ generated by C and all the standard matrix units e_{ij} of $M_\omega(F)$. Note that

$$D = C + \text{socle } B(F)$$

where socle $B(F)$ consists of all $\omega \times \omega$ matrices with only finitely many nonzero entries. Now Propositions 5.1 and 5.4 hold with A replaced by D, provided we interpret "generators" as the above generators of C. (D itself is not finitely generated although it is countable-dimensional.) The only use we make of the extra matrix units in D is to get the above e_{ji} (in logarithmic time) as a cross-element from fD to gD; we have $g = e_{jj} = e_{ji}e_{ij} \in e_{ji}D$ and $f = e_{ii} = e_{ij}e_{ji} \in De_{ji}$ so $gD = e_{ji}D$ and $Df = De_{ji}$. The proof in Section 7 now carries over and shows that D has bandwidth dimension r.

It remains to show that D is directly finite. Observe that C is a subalgebra of the algebra L of all block lower triangular matrices relative to a scheme II diagonal. Since the diagonal blocks are all directly finite, it follows that L is directly finite. Suppose $\alpha, \beta \in D$ with $\alpha\beta = 1$. Since $D = C +$ socle $B(F)$, we can write

$$\alpha = \left[\begin{array}{cc} \alpha_1 & 0 \\ \hline \alpha_3 & \alpha_2 \end{array}\right], \qquad \beta = \left[\begin{array}{cc} \beta_1 & 0 \\ \hline \beta_3 & \beta_2 \end{array}\right]$$

where the top left hand blocks are finite $n \times n$ for the same n, and where

$$\left[\begin{array}{cc} 0 & 0 \\ \hline 0 & \alpha_2 \end{array}\right], \quad \left[\begin{array}{cc} 0 & 0 \\ \hline 0 & \beta_2 \end{array}\right] \in L.$$

Then $\alpha_1\beta_1 = 1$ and $\alpha_2\beta_2 = 1$, so by the direct finiteness of $M_n(F)$ and L we have $\beta_1\alpha_1 = 1$ and $\beta_2\alpha_2 = 1$. Hence α has zero right annihilator in D, whence from $\alpha(1 - \beta\alpha) = 0$ we infer that $1 - \beta\alpha = 0$ and so $\beta\alpha = 1$. Therefore D is a directly finite, countable-dimensional algebra of bandwidth dimension r.

For the case $r = 1$, we can replace the spine S of Section 1 by the spine determined by the sequence

$$2^1, 2^2, 2^2, 2^3, 2^3, 2^3, \ldots, 2^m, 2^m, \ldots, 2^m \ (m \text{ lots}), \ldots$$

and modify the generators accordingly. If the resulting algebra $D = C +$ socle $B(F)$ embeds in $G(s)$ for some $s < 1$, then Step 5 in Section 7 shows that the kth term, n_k, of the above sequence must have at most polynomial growth. This contradicts the fact that n_k has exponential growth ($n_k \geq 2^{\sqrt{k}}$). Thus D has bandwidth dimension 1 and is directly finite and countable-dimensional. ∎

Remark. It seems likely that there are in fact finitely generated algebras which are directly finite and of any prescribed bandwidth dimension $r \in [0, 1]$.

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